HIGHER ORDER NON-HOMOGENOUS DIFFERENTIAL EQUATIONS

Complete the following problems to reinforce your understanding of the concept covered in this module.

Problem 1:

Solve the following differential equation:

$$y'' - 2y' - 3y = e^{2t}$$

Problem 2:

Solve the following differential equation:

$$y'' - 2y' - 3y = 3t^2 + 4t - 5$$

Problem 3:

Solve the following differential equation:

$$y'' - 2y' - 3y = 5\cos(2t)$$

HIGHER ORDER NON-HOMOGENOUS DIFFERENTIAL EQUATIONS

Solution 1:

The first step is to determine the corresponding homogeneous solution to:

$$y'' - 2y' - 3y = 0.$$

The characteristic equation is:

$r^2 - 2r - 3 = 0$ which factors out to $(r+1)(r-3) = 0$.

The roots are r=-1 and r=3, which shows that we have two real distinct roots, make the corresponding homogeneous solution:

$$y_h = c_1 e^{-t} + c_2 e^{3t}$$

The nonhomogeneous equation has $g(t) = e^{2t}$. We know that an exponential function does not change form after differentiation, therefore, we can reasonably expect that Y_P is in the form Ae^{2t} where A is some coefficient that we will determine through using the Method of Undetermined Coefficients

So let's guess that the nonhomogeneous solution is $Y_P = Ae^{2t}$.

Differentiating we find that:

$$y'(t) = 2Ae^{2t}$$
$$y''(t) = 4Ae^{2t}$$

Substituting these values back in to the original equation we get:

$$4Ae^{2t} - 2(2Ae^{2t}) - 3(Ae^{2t}) = e^{2t}$$

Now we set the 'like' coefficients from each side against each other and solve, we get:

$$-3A = 1$$
$$A = -\frac{1}{3}$$

Since we were able to determine a value for A, that means our guess was correct and our nonhomogeneous solution is:

$$Y_p = -\frac{1}{3}e^{2t}$$

The overall solution is then:

$$y(t) = y_h + Y_p = c_1e^{-t} + c_2e^{3t} - \frac{1}{3}e^{2t}$$

Note: When an exponential function appears in $g(t)$, use an exponential function of the same exponent for y_{nh}

Solution 2:

The first step is to determine the corresponding homogeneous solution to:

$$y'' - 2y' - 3y = 0.$$

The characteristic equation is:

$r^2 - 2r - 3 = 0$ which factors out to $(r+1)(r-3) = 0$.

The roots are r=-1 and r=3, which shows that we have two real distinct roots, make the corresponding homogeneous solution:

$$y_h = c_1 e^{-t} + c_2 e^{3t}$$

The nonhomogeneous equation has $g(t) = 3t^2 + 4t - 5$, a 2nd degree (quadratic) polynomial. Polynomials, like exponential functions, do not change form after differentiation. The derivative of a polynomial is just another polynomial of one degree less; until it eventually reaches zero. We can reasonably expect that Y_p will be in the form of a polynomial as well.

So, we will let Y_p be a general quadratic polynomial $At^2 + Bt + C$, where A, B, and C are some coefficients that we will determine through using the Method of Undetermined Coefficients

So let's guess that the nonhomogeneous solution is $Y_p = At^2 + Bt + C$.

Differentiating we find that:

$$y'(t) = 2At + B$$
$$y''(t) = 2A$$

Substituting these values back in to the original equation we get:

$$(2A) - 2(2At + B) - 3(At^2 + Bt + C) = 3t^2 + 4t - 5$$
$$-3At^2 + (-4A - 3B)t + (2A - 2B - 3C) = 3t^2 + 4t - 5$$

Now we set the 'like' coefficients from each side against each other and solve for the unknowns to get:

t^2: $\quad -3A = 3 \qquad A = -1$

t: $\quad -4A - 3B = 4 \qquad B = 0$

1: $\quad 2A - 2B - 3C = -5 \qquad C = 1$

Since we were able to determine values for A, B, and C, that means our guess was correct and our nonhomogeneous solution is:

$$Y_P = -t^2 + 1$$

The overall solution is then:

$$y(t) = y_h + Y_P = c_1 e^{-t} + c_2 e^{3t} - t^2 + 1$$

Note: When a polynomial appears in $g(t)$, use a general polynomial of the same degree for Y_P. If $g(t)$ is a constant, it is considered a polynomial of degree 0, and Y_P would also be a general polynomial of degree 0, or a constant.

Solution 3:

The first step is to determine the corresponding homogeneous solution to:

$$y'' - 2y' - 3y = 0.$$

The characteristic equation is:

$r^2 - 2r - 3 = 0$ which factors out to $(r+1)(r-3) = 0$.

The roots are r=-1 and r=3, which shows that we have two real distinct roots, make the corresponding homogeneous solution:

$$y_h = c_1 e^{-t} + c_2 e^{3t}$$

The nonhomogeneous equation has $g(t) = 5\cos(2t)$. The derivative of a trig function does change form when differentiated, although predicatively. Cosines change to sines and visa versa, therefore, we can reasonably expect that Y_P will be in the form that includes both a sine and cosine.

So, we will let Y_p be a general trig solution $A\cos(2t)+B\sin(2t)$, where A and B are some coefficients that we will determine through using the Method of Undetermined Coefficients

So let’s guess that the nonhomogeneous solution is $Y_p = A\cos(2t)+B\sin(2t)$.

Differentiating we find that:

$$y'(t) = -2A\sin(2t)+2B\cos(2t)$$
$$y''(t) = -4A\cos(2t)-4B\sin(2t)$$

Substituting these values back in to the original equation we get:

$$(-4A\cos(2t)-4B\sin(2t))-2(-2A\sin(2t)+2B\cos(2t))-3(A\cos(2t)+B\sin(2t)) = 5\cos(2t)$$

$$(-7A-4B)\cos(2t)+(4A-7B)\sin(2t) = 5\cos(2t)$$

Now we set the ‘like’ coefficients from each side against each other and solve for the unknowns to get:

$$\cos(2t): \quad -7A-4B = 5$$
$$\sin(2t): \quad 4A-7B = 0$$

We have two equations and two unknowns, so solving for A and B we get:

$$A = -\frac{7}{13}$$
$$B = -\frac{4}{13}$$

Since we were able to determine values for A and B, that means our guess was correct and our nonhomogeneous solution is:

$$Y_p = -\frac{7}{13}\cos(2t) - \frac{4}{13}\sin(2t)$$

The overall solution is then:

$$y(t) = y_h + Y_p = c_1 e^{-t} + c_2 e^{3t} - \frac{7}{13}\cos(2t) - \frac{4}{13}\sin(2t)$$

Note: When sine or cosines appear in $g(t)$, use a general trig solution of using both sine and cosine as Y_p.