TAYLOR AND MACLAURIN SERIES

Complete the following problems to reinforce your understanding of the concept covered in this module.

Problem 1:

Determine the Maclaurin series for $f(x) = \sin(\pi x)$

Problem 2:

Determine the Taylor series for $f(x) = xe^x$ about x=0

Problem 3:

Find the fourth Maclaurin polynomial for $f(x) = \sin(2x)$

TAYLOR AND MACLAURIN SERIES

Solution 1:

Recall that the Maclaurin Series for $f(x)$ is generally represented as:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(0)}{n!}x^{n}=f(0)+f'(0)x+\frac{f''(0)}{2!}x^{2}+\frac{f'''(0)}{3!}x^{3}+\ldots$$

To find the Maclaurin Series for a function the general formula for $f^{n}(a)$ must be defined and then plugged in to the general formula of the Maclaurin Series. The general formula $f^{n}(a)$ can be derived as:

$$f(a)=\sin(\pi x) \qquad f(0)=0$$

$$f'(a)=\pi\cos(\pi x) \qquad f'(0)=\pi$$

$$f''(a)=-\pi^{2}\sin(\pi x) \qquad f''(0)=0$$

$$f'''(a)=-\pi^{3}\cos(\pi x) \qquad f'''(0)=-\pi^{3}$$

$$f''''(a)=\pi^{4}\sin(\pi x) \qquad f''''(0)=0$$

There is not an easy formula to derive for either the general derivative or the evaluation of the derivative. However, there is a clear pattern to the values. So, let’s plug what we’ve got into the series:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(0)}{n!}x^{n}=0+\pi x+\frac{0}{2!}x^{2}+\frac{-\pi^{3}}{3!}x^{3}+\ldots$$

Dropping the zero terms we get:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{n}(0)}{n!}x^{n}=\pi x+\frac{-\pi^{3}}{3!}x^{3}+\frac{\pi^{5}}{5!}x^{5}$$

So the Maclaurin series is:

$$\sin(\pi x)=\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}x^{2n+1}$$

Solution 2:

A Taylor series about $x=0$ is called a Maclaurin Series for $f(x)$, generally represented as:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$$

To find the Taylor Series for a function the general formula for $f^n(a)$ must be defined and then plugged in to the general formula.

The general formula $f^n(a)$ can be derived as:

$$f(a)=xe^x \qquad f(0)=0$$

$$f'(a)=e^x+xe^x \qquad f'(0)=1+0=1$$

$$f''(a)=e^x+e^x+xe^x \qquad f''(0)=1+1+0=2$$

$$f'''(a)=e^x+e^x+e^x+xe^x \qquad f'''(0)=1+1+1+0=3$$

$$f''''(a)=e^x+e^x+e^x+e^x+xe^x \qquad f''''(0)=1+1+1+1+0=4$$

$$f^n(0)=n$$

The Taylor series about x=0 for $f(x)=\text{xe}^x$ is:

$$\text{xe}^x=\sum_{n=0}^{\infty}\frac{n}{n!}x^n$$

Solution 3:

A Maclaurin series is a Taylor series about x=0.

To find the fourth term of the series, first find the first four $f^n(0)$ terms and then plug them in to the general equation:

$$f(x)=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots$$

So, the fourth term is n=4, so we need terms for n=0 through n=4, which are:

$$f(a)=\sin(2x) \qquad f(0)=0$$

$$f'(a)=2\cos(2x) \qquad f'(0)=2$$

$$f''(a)=-4\sin(2x) \qquad f''(0)=0$$

$$f'''(a)=-8\cos(2x) \qquad f'''(0)=-8$$

$$f''''(a)=16\sin(2x) \qquad f''''(0)=0$$

The fourth term is then:

$$\sin(2x)=0+\frac{2}{1!}x+\frac{0}{2!}x^2+\frac{-8}{3!}x^3+\frac{0}{4!}x^4=2x-\frac{4}{3}x^3$$