CROSS PRODUCT

Given two vectors $\vec{a} = < a_1, a_2, a_3 >$ and $\vec{b} = < b_1, b_2, b_3 >$, the cross product is found using the following formula:

$$\vec{a} \times \vec{b} = < a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 >$$

This formula can be derived quickly for each unique scenario by first inserting the two vectors in to a 3x3 matrix such that:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Where the first row is the standard basis vectors and must appear in the order given, the second row is the components of $\vec{a}$ and the third row is the components of $\vec{b}$.

Use the Method of Cofactors to determine the cross product formula from this point. This method can is illustrated as:

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \text{ where } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

It is important to note that in the Method of Cofactors, the terms alternate in sign and each individual is simply a 2x2 matrix multiplied by a scalar.

The cross product can also be illustrated geometrically. Given two vectors $\vec{a} = < a_1, a_2, a_3 >$ and $\vec{b} = < b_1, b_2, b_3 >$ with an angle Θ between them, where Θ is between 0 and π, the cross product can be found using the following formula:

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\left|\vec{b}\right| \sin\Theta$$

Concept Example:

The following problem introduces the concept reviewed within this module. Use this content as a primer for the subsequent material.

Given the two vectors $\vec{a} = <2,1,-1>$ and $\vec{b} = <-3,4,1>$ determine the cross product $\vec{a} \times \vec{b}$

Solution:

The cross product can be found by first inserting the two vectors in to a 3x3 matrix such that:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ -3 & 4 & 1 \end{vmatrix}$$

Using the Method of Cofactors, the cross product is:

$$\vec{a} \times \vec{b} = \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} \vec{k} \text{ where } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The cross product of the two vectors is then:

$$\vec{a} \times \vec{b} = 5\vec{i} - \vec{j} + 11\vec{k}$$