MATRIX ECHELON FORM

A matrix is considered to be in row echelon form A_{ref} when it adheres to the following conditions:

- The first non-zero element in each row, called the leading entry, is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having a non-zero element.

A matrix is considered to be in reduced row echelon form A_{rref} when it satisfies the following conditions:

- The matrix is in row echelon form (satisfies three conditions above).
- The leading entry in each row is the only non-zero entry in its column.

Any matrix can be reduced into its echelon forms, using a series of elementary row operations, following the general procedure:

1. Pivot the matrix:

    1. Find the pivot, the first non-zero entry in the first column of the matrix.
    2. Interchange rows, moving the pivot row to the first row.
    3. Multiply each element in the pivot row by the inverse of the pivot, so the pivot equals 1.
    4. Add multiples of the pivot row to each of the lower rows, so every element in the pivot column of the lower rows equals 0.

2. To get the matrix in row echelon form, A_{ref}, repeat the pivot:

    1. Repeat the procedure from Step 1 above, ignoring previous pivot rows.
    2. Continue until there are no more pivots to be processed.

3. To get the matrix in reduced row echelon form, A_{rref}, process non-zero entries above each pivot.

    1. Identify the last row having a pivot equal to 1, and let this be the pivot row.
    2. Add multiples of the pivot row to each of the upper rows, until every element above the pivot equals 0.
    3. Moving up the matrix, repeat this process for each row.

Concept Example:

The following problem introduces the concept reviewed within this module. Use this content as a primer for the subsequent material.

Complete row operations on the matrix below to convert it to its reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

Solution:

Recall that any matrix can be reduced into its echelon forms, using a series of elementary row operations, following the general procedure:

1. Pivot the matrix:

    1. Find the pivot, the first non-zero entry in the first column of the matrix.
    2. Interchange rows, moving the pivot row to the first row.
    3. Multiply each element in the pivot row by the inverse of the pivot, so the pivot equals 1.

4. Add multiples of the pivot row to each of the lower rows, so every element in the pivot column of the lower rows equals 0.

Let's work on the first column:

Multiply Row 1 by 2 and add to row 2 ($R2+2R1$)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix}$$

Multiply Row 1 by -3 and add to row 3 ($R3-3R1$)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

The first pivot column is complete, so let's repeat the procedure from Step 1 above, ignoring previous pivot rows and continue until there are no more pivots to be processed.

Multiply Row 2 by 1 and add to row 3 ($R3+R2$)

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in Row Echelon Form

Recall, to get a matrix in reduced row echelon form, A_{rref}, process non-zero entries above each pivot, the process is:

1. Identify the last row having a pivot equal to 1, and let this be the pivot row.

2. Add multiples of the pivot row to each of the upper rows, until every element above the pivot equals 0.
3. Moving up the matrix, repeat this process for each row.

In this case, the last pivot equal to one is in the second row, second column. Completing the process, we:

Multiply Row 2 by -2 and add to row 1 ($R1 - 2R2$)

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in Reduced Row Echelon Form