HIGHER ORDER NON-HOMOGENOUS DIFFERENTIAL EQUATIONS

When given an equation in the form:

$$y'' + p(t)y' + q(t)y = g(t)$$

Use the following process to determine the general solution:

The first step is to determine the corresponding homogeneous solution to:

$$y'' + p(t)y' + q(t)y = 0$$

Next, determine the non-homogeneous particular solution, Y_P, by using the Method of Undetermined Coefficients. This method is based on a guessing technique where, Y_P will be guessed after observing the form of $g(t)$. This guess is then differentiated so that we have:

$$y(t)$$
$$y'(t)$$
$$y''(t)$$

The values are substituted back in to the original equation and the like terms are combined.

The 'like' coefficients from each side are set against each other and each unknown coefficient is then solved for. If values are able to be determined, then the original guess was correct, if not, the original guess was wrong.

Combine the homogeneous and non-homogeneous solutions to obtain the overall solution of the differential equation.

$$y(t) = y_h + Y_p$$

Concept Example:

The following problem introduces the concept reviewed within this module. Use this content as a primer for the subsequent material.

Solve the following differential equation:

$$y'' + 2y' + 5y = \cos(3t)$$

Solution:

The first step is to determine the corresponding homogeneous solution to:

$$y'' + 2y' + 5y = 0.$$

The characteristic equation is:

$r^2 + 2r + 5 = 0$ which by using the quadratic formula we can find that the roots are $r_{1,2} = \dfrac{-2 \pm \sqrt{-16}}{2}$ or $r_{1,2} = -1 \pm 2i$, two complex roots, making the corresponding homogeneous solution:

$$y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$$

The nonhomogeneous equation has $g(t) = \cos(3t)$. The derivative of a trig function does change form when differentiated, although predicatively. Cosines change to sines and visa versa, therefore, we can reasonably expect that Y_p will be in the form that includes both a sine and cosine.

So lets guess that $y(t) = A\cos(3t) + B\sin(3t)$ is the solution.

Differentiating we find that:

$$y'(t) = -3A\sin(3t) + 3B\cos(3t)$$
$$y''(t) = -9A\cos(3t) - 9B\sin(3t)$$

Substituting these values back in to the original equation we get:

$$-9A\cos(3t) - 9B\sin(3t) + 2(-3A\sin(3t) + 3B\cos(3t)) + 5(A\cos(3t) + B\sin(3t)) = \cos(3t)$$
$$(-4A + 6B)\cos(3t) + (-4B + 6A)\sin(3t) = \cos(3t)$$

Now we set the 'like' coefficients from each side against each other and solve, we get:

$$\cos(3t): \quad -4A + 6B = 1$$
$$\sin(3t): \quad -4B + 6A = 0$$

With two equations and two unknowns, we solve for A and B such that:

$$A = -\frac{1}{13}$$
$$B = \frac{3}{26}$$

Since we were able to determine values for A and B, that means are guess was right and our nonhomogeneous solution is:

$$Y_p = -\frac{1}{13}\cos(3t) + \frac{3}{26}\sin(3t)$$

and the overall solution is:

$$y = y_h + Y_p = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t - \frac{1}{13}\cos(3t) + \frac{3}{26}\sin(3t)$$