When asked to integrate a rational expression, it is typical to look for the numerator to be a derivative of the denominator and then simply substitute and solve. However, when this is not possible, a process known as partial fraction decomposition must be employed, which consists of decomposing the rational expression into simpler rational expressions that can then be added or subtracted.

Working with a rational expression in the form:

$$f(x)=\frac{p(x)}{q(x)}$$

where p*(x)* and *q(x)* are polynomials and the degree of *p(x)* is smaller than the degree of q*(x)*. Partial fractions can only be done if the degree of the numerator is less than the degree of the denominator.

Once it is determined that partial fractions can be completed on the rational function, the denominator is factored as much as possible in to an equivalent expression in using multiple terms determined by the factor in the original denominator.

Factor in denominator	Term(s) in Partial Fraction Decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Once the rational expression is broken down in to Partial Fractions, the next step in solving is to set the numerators from each side equal to one another and solving for the unknown values.

Concept Example:

The following problem introduces the concept reviewed within this module. Use this content as a primer for the subsequent material.

Evaluate the integral:

$$\int \frac{x^2 - x + 1}{(x+1)^3} dx$$

Solution:

The first step is to factor the denominator as much as possible, which it is in this case, and get the terms of the partial fraction decomposition using the table:

$$\frac{x^2 - x + 1}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

The next step is to add the right side back up by setting the terms together with a common denominator:

$$\frac{x^2 - x + 1}{(x+1)^3} = \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3}$$

Now, choose *A*, *B,* and *C* so that the numerators on each side are equal for every *x*. To do this, set the numerators equal:

$$x^2 - x + 1 = A(x+1)^2 + B(x+1) + C$$

Look for values that make one of the unknowns equal to zero and solve for the other.

$$x = -1: \qquad -1^2 + 1 + 1 = A(-1+1)^2 + B(-1+1) + C \Rightarrow C = 3$$

$$x = 0: \qquad 0^2 - 0 + 1 = A(0+1)^2 + B(0+1) + 3 \Rightarrow A + B = -2$$

$$x = 1: \qquad 1^2 - 1 + 1 = A(1+1)^2 + B(1+1) + 1 \Rightarrow 4A + 2B = -1$$

Two equations and two unknowns left, so $A = 1$ and $B = -3$

Plugging these values back in to the original decomposed function and carry out the integration.

$$\int \frac{1}{(x+1)} + \frac{-3}{(x+1)^2} + \frac{3}{(x+1)^3} dx$$

So the solution is:

$$\int \frac{x^2 - x + 1}{(x+1)^3} dx = \ln(x+1) + \frac{3}{(x+1)} - \frac{3}{2(x+1)^2} + c$$