PARTIAL FRACTIONS

Complete the following problems to reinforce your understanding of the concept covered in this module.

Problem 1:

Integrate the following function:

$$f(x) = \frac{1}{x^2 - 4}$$

Problem 2:

Evaluate the integral:

$$\int \frac{x^2 + x - 1}{x(x^2 - 1)} dx$$

Problem 3:

Evaluate the integral:

$$\int \frac{x + 7}{x^2(x + 2)} dx$$

PARTIAL FRACTIONS

Solution 1:

Integrating the function:

$$\int \frac{1}{x^2 - 4} dx$$

The first step is to factor the denominator as much as possible and get the terms of the partial fraction decomposition using the table:

$$\frac{1}{(x+2)(x-2)} = \frac{A}{(x+2)} + \frac{B}{(x-2)}$$

The next step is to add the right side back up by setting the terms together with a common denominator:

$$\frac{1}{(x+2)(x-2)} = \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$$

Now, choose *A* and *B* so that the numerators on each side are equal for every *x*. To do this, set the numerators equal:

$$1 = A(x-2) + B(x+2)$$

Look for values that make one of the unknowns equal to zero and solve for the other.

$$x = -2: \qquad 1 = A(-2-2) + B(-2+2) \Rightarrow A = -\frac{1}{4}$$

$$x = 2: \qquad 1 = A(2-2) + B(2+2) \Rightarrow B = \frac{1}{4}$$

Plugging these values back in to the original decomposed function and carry out the integration.

$$\int\left(\frac{-\frac{1}{4}}{(x+2)}+\frac{\frac{1}{4}}{(x-2)}\right)dx$$

$$-\frac{1}{4}\ln(x+2)+\frac{1}{4}\ln(x-2)+c$$

Recall that $\ln m-\ln n=\ln\frac{m}{n}$, so the solution is:

$$\int\frac{1}{x^2-4}dx=-\frac{1}{4}\ln\frac{x+2}{x-2}$$

Solution 2:

The first step is to factor the denominator as much as possible and get the terms of the partial fraction decomposition using the table:

$$\frac{x^2+x-1}{x(x^2-1)}=\frac{x^2+x-1}{x(x+1)(x-1)}=\frac{A}{x}+\frac{B}{(x+1)}+\frac{C}{(x-1)}$$

The next step is to add the right side back up by setting the terms together with a common denominator:

$$\frac{x^2+x-1}{x(x+1)(x-1)}=\frac{A(x+1)(x-1)+Bx(x-1)+Cx(x+1)}{x(x+1)(x-1)}$$

Now, choose *A*, *B*, and *C* so that the numerators on each side are equal for every *x*. To do this, set the numerators equal:

$$x^2+x-1=A(x+1)(x-1)+Bx(x-1)+Cx(x+1)$$

Look for values that make one of the unknowns equal to zero and solve for the others.

$x=0$: $\quad 0^2+0-1=A(0+1)(0-1)+B(0)(0-1)+C(0)(0+1)\Rightarrow A=1$

$$x=-1: \qquad -1^2-1-1=A(-1+1)(-1-1)+B(-1)(-1-1)+C(-1)(-1+1) \Rightarrow B=-\frac{1}{2}$$

$$x=1: \qquad 1^2+1-1=A(1+1)(1-1)+B(1)(1-1)+C(1)(1+1) \Rightarrow C=\frac{1}{2}$$

Plugging these values back in to the original decomposed function and carry out the integration.

$$\int \frac{1}{x}+\frac{-\frac{1}{2}}{(x+1)}+\frac{\frac{1}{2}}{(x-1)}dx$$

$$\ln(x)-\frac{1}{2}\ln(x+1)+\frac{1}{2}\ln(x-1)+c$$

Recall that $\ln m-\ln n=\ln\frac{m}{n}$, so the solution is:

$$\int \frac{x^2+x-1}{x(x^2-1)}dx=\ln(x)+\frac{1}{2}\ln\frac{(x-1)}{(x+1)}+c$$

Solution 3:

The first step is to factor the denominator as much as possible, which in this case it already is, and get the terms of the partial fraction decomposition using the table.

$$\frac{x+7}{x^2(x+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{(x+2)}$$

The next step is to add the right side back up by setting the terms together with a common denominator:

$$\frac{x+7}{x^2(x+2)}=\frac{A(x)(x+2)+B(x+2)+C(x^2)}{x^2(x+2)}$$

Now, choose *A*, *B*, and *C* so that the numerators on each side are equal for every *x*. To do this, set the numerators equal:

$$x+7=A(x)(x+2)+B(x+2)+C(x^2)$$

Look for values that make one of the unknowns equal to zero and solve for the others.

$x=0$: $$0+7=A(0)(0+2)+B(0+2)+C(0^2)\Rightarrow B=\frac{7}{2}$$

$x=-1$: $$-1+7=A(-1)(-1+2)+B(-1+2)+C(-1^2)\Rightarrow C=\frac{5}{4}$$

$x=1$: $$1+7=A(1)(1+2)+B(1+2)+C(1^2)\Rightarrow A=-\frac{5}{4}$$

Plugging these values back in to the original decomposed function and carry out the integration.

$$\int \frac{-\frac{5}{4}}{x}+\frac{\frac{7}{2}}{x^2}-\frac{\frac{5}{4}}{(x+2)}dx$$

$$-\frac{5}{4}\ln(x)-\frac{7}{2x}+\frac{5}{4}\ln(x+2)+c$$

So the solution is:

$$\int \frac{x+7}{x^2(x+2)}dx=-\frac{5}{4}\ln(x)-\frac{7}{2x}+\frac{5}{4}\ln(x+2)+c$$