L'HOPITAL'S RULE

Complete the following problems to reinforce your understanding of the concept covered in this module.

Problem 1:

Evaluate $\lim_{x \to \infty} \frac{x^3 - 4x^2 + 1}{3x^3 - 1}$

Problem 2:

Evaluate $\lim_{x \to 0} x \cot x$

Problem 3:

Evaluate $\lim_{x \to \infty} x\left(e^{\frac{1}{x}} - 1\right)$

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Solution 1:

Using normal procedures and taking the limit of the function results in an indeterminate form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{x^3-4x^2+1}{3x^3-1}=\frac{\infty}{\infty}$$

L' Hôpital's Rule can be used to assist in determining the actual limit, recall that:

If the $\lim f(x)$ and $\lim g(x)$ are both infinite, then:

- If $\lim\frac{f'(x)}{g'(x)}=L$, then $\lim\frac{f(x)}{g(x)}=\lim\frac{f'(x)}{g'(x)}+L$
- If $\lim\frac{f'(x)}{g'(x)}$ tends to $+\infty$ or $-\infty$, then so does $\frac{f(x)}{g(x)}$

Therefore, carrying out L' Hôpital's Rule gives:

$$\lim_{x\to\infty}\frac{x^3-4x^2+1}{3x^3-1}=\lim_{x\to\infty}\frac{3x^2-8x}{9x^2}=\frac{\infty}{\infty}$$

The result still renders a limit of the indeterminate form $\frac{\infty}{\infty}$, therefore, L' Hôpital's Rule can be applied until a limit is found:

$$\lim_{x\to\infty}\frac{3x^2-8x}{9x^2}=\lim_{x\to\infty}\frac{6x-8}{18x}=\lim_{x\to\infty}\frac{6}{18}=\frac{1}{3}$$

Solution 2:

Taking the limit of the function results in $0 \cdot \infty$.

$$\lim_{x \to 0} x \cot x = 0 \cdot \infty$$

This isn't an indeterminate form, nor does it tell us anything about the limit, so convert the x cot x to a form that will either render $\frac{\infty}{\infty}$ or $\frac{0}{0}$. Recall that $\cot x = \frac{\cos x}{\sin x}$, therefore the problem can be rewritten in the following form:

$$\lim_{x \to 0} x \cot x = \lim_{x \to 0} x \frac{\cos x}{\sin x} = \lim_{x \to 0} \frac{x \cos x}{\sin x}$$

Evaluating the limit:

$$\lim_{x \to 0} \frac{x \cos x}{\sin x} = \frac{0}{0}$$

Using L' Hôpital's Rule:

$$\lim_{x \to 0} \frac{1 \cdot \cos x - x \sin x}{\cos x} = \frac{1-0}{1} = 1$$

Solution 3:

As x approaches infinite, the limit evaluates to $0 \cdot \infty$.

$$\lim_{x \to \infty} x \left(e^{\frac{1}{x}} - 1 \right) = \infty \cdot 0$$

This isn't an indeterminate form, nor does it tell us anything about the limit, so convert the equation to a form that will either render $\frac{\infty}{\infty}$ or $\frac{0}{0}$. To do this, use substitutions:

Let $u = \frac{1}{x}$ and then $x = \frac{1}{u}$ so as $x \to \infty$, $u \to 0$

Rewriting:

$$\lim_{x\to\infty} x\left(e^{\frac{1}{x}} - 1\right) = \lim_{u\to 0}\frac{1}{u}\left(e^{u} - 1\right) = \lim_{u\to 0}\frac{\left(e^{u} - 1\right)}{u} = \frac{0}{0}$$

Using L' Hôpital's Rule:

$$\lim_{u\to 0}\frac{e^{u}}{1} = 1$$