PARTIAL DERIVATIVES

When given a function of two variables, the partial derivative of that function can be found by concentrating exclusively on changing one of the variables at a time, while holding the remaining variables fixed and treating them like a constant.

So in essence, determining the derivatives of functions of more than one variable is done in pretty much the same way as taking derivatives of a single variable.

To compute:

- The partial derivative of $f(x, y)$ with respect to x, treat all the *y*'s as constants and then differentiate the *x*'s.
- The partial derivative of $f(x, y)$ with respect to y, treat all the *x*'s as constants and then differentiate the *y*'s as we are used to doing.

These two partial derivatives are often referred to as first order partial derivatives. Just as with functions of one variable, derivatives of all orders, as long as they exist, are possible.

It's important to note that the notation for partial derivatives is different than that for derivatives of functions of a single variable. With functions of a single variable a single prime is used to denote a derivative. With partial derivatives we will always need to remember the variable that we are differentiating with respect to and so a variable subscript is often used to define that characteristic.

The formal definitions of the two variable partial derivatives are:

$$f_x(x, y) = \lim_{h \to o}\left(\frac{f(x+h, y) - f(x, y)}{h}\right)$$

$$f_y(x,y) = \lim_{h \to o}\left(\frac{f(x,y+h) - f(x,y)}{h}\right)$$

Some possible equivalent notations for partial derivatives that may be encountered are:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = D_y f$$

Concept Example:

The following problem introduces the concept reviewed within this module. Use this content as a primer for the subsequent material.

Determine the partial derivatives of the following function:

$$f(x,y) = \frac{x-y}{x+y}$$

Solution:

Let's start with determining the partial derivative with respect to x. Recall that to find the partial derivative of $f(x,y)$ with respect to x, we need to treat all the *y*'s as constants and then differentiate the *x*'s.

The original function is:

$$f(x,y) = \frac{x-y}{x+y}$$

Recall that the Quotient Rule states:

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Therefore:

$$f(x) = (x - y) \quad \text{and} \quad g(x) = (x + y)$$
$$f'(x) = 1 \quad \text{and} \quad g'(x) = 1$$

Plugging in the values we find that:

$$\frac{\partial f}{\partial x} = \frac{(x+y)1 - (x-y)1}{(x+y)^2} = \frac{x + y - x + y}{(x+y)^2}$$

So:

$$\frac{\partial f}{\partial x} = \frac{2y}{(x+y)^2}$$

Repeating the process for the partial derivative with respect to y:

Using that the Quotient Rule:

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Therefore:

$$f(y) = (x - y) \quad \text{and} \quad g(y) = (x + y)$$
$$f'(y) = -1 \quad \text{and} \quad g'(y) = 1$$

Plugging in the values we find that:

$$\frac{\partial f}{\partial y} = \frac{(x+y)(-1)-(x-y)(1)}{(x+y)^2} = \frac{-x-y-x+y)}{(x+y)^2}$$

So:

$$\frac{\partial f}{\partial y} = -\frac{2x}{(x+y)^2}$$