CRITICAL POINTS

Complete the following problems to reinforce your understanding of the concept covered in this module.

Problem 1:

To find the maxima and minima of the function:

$f(x) = x^2 - 5x + 7$ in the interval $-1 \leq x \leq 3$

Problem 2:

Determine the critical points of the function:

$$f(x) = x^3 + 3x^2 - 24x + 3$$

Problem 3:

Find the local extrema of the function:

$$f(x) = \sin(x) + \cos(x)$$

psst…don't forget to take notes

CRITICAL POINTS

Solution 1:

To find the maxima and minima of the function $f(x) = x^2 - 5x + 7$ between the interval $-1 \leq x \leq 3$ we first need to take the derivative, such that:

$$f'(x) = 2x - 5$$

With this function, it is said that at x=c there is a critical point if either of the following scenarios hold true:

$f'(c) = 0$
$f'(c)$ doesn't exist

We know that the function exists all along the interval, so focus on the roots, $f'(c) = 0$ to determine the critical points. Setting the derivative against zero we get:

$$2x - 5 = 0$$
$$x = 2.5$$

This point falls within the range $-1 \leq x \leq 3$, so this is determined to be the critical point.

Now to determine whether this point is a local maximum or minimum, the Second Derivative Test is used, which states:

- If $f''(c) > 0$, then $f(x)$ is increasing in the interval around c. Since $f'(c) = 0$, then $f(x)$ must be negative to the left of c and positive to the right of c. Therefore, c is a local minimum.

- If $f''(c)<0$, then $f(x)$ is decreasing in the interval around *c*. Since $f'(c)=0$, then $f(x)$ must be positive to the left of *c* and negative to the right of *c*. Therefore, *c* is a local maximum.

The Second derivative of $f(x)$ is:

$$f''(x) = 2$$

Since 2>0, there is a local minimum at $x = 2.5$

Solution 2:

To find the critical points of the function $f(x) = x^3 + 3x^2 - 24x + 3$, we first need to take the derivative, such that:

$$f'(x) = 3x^2 + 6x - 24$$

With this function, it is said that at x=c there is a critical point if either of the following scenarios hold true:

$f'(c)=0$
$f'(c)$ doesn't exist

We know that the function exists for all points, so focus on the roots, $f'(c)=0$ to determine the critical points. Setting the derivative against zero we get:

$$3x^2 + 6x - 24 = 0$$
$$x^2 + 2x - 8 = 0$$
$$(x-2)(x+4) = 0$$

The critical points are located at $x = 2$ and $x = -4$

Solution 3:

Since $\sin(x)$ and $\cos(x)$ are continuous and differentiable everywhere, then $f(x)$ is continuous and differentiable everywhere. So the critical points of $f(x)$ are going to be the roots of the derivative:

$$f'(x) = \cos(x) - \sin(x)$$

Setting this function against zero we get:

$$\cos(x) - \sin(x) = 0$$

Solving we find that :

$$\cos(x) = sin(x)$$

Trigonometric algebra tells us that roots occur at $x = \frac{\pi}{4} + 2n\pi$ and $x = \frac{5\pi}{4} + 2n\pi$, where $n = 0, \pm 1, \pm 2$

Now to determine whether these points are local maximums or minimums, the Second Derivative Test is used, which states:

- If $f''(c) > 0$, then $f(x)$ is increasing in the interval around c. Since $f'(c) = 0$, then $f(x)$ must be negative to the left of c and positive to the right of c. Therefore, c is a local minimum.

- If $f''(c) < 0$, then $f(x)$ is decreasing in the interval around c. Since $f'(c) = 0$, then $f(x)$ must be positive to the left of c and negative to the right of c. Therefore, c is a local maximum.

The Second derivative of $f(x)$ is:

$$f''(x) = -\sin(x) - \cos(x)$$

Plugging in the critical point:

$$f''\left(\frac{\pi}{4}+2n\pi\right)=-2\frac{\sqrt{2}}{2}=-\sqrt{2}$$

$$f''\left(\frac{5\pi}{4}+2n\pi\right)=2\frac{\sqrt{2}}{2}=\sqrt{2}$$

The second derivative test concludes that $x=\frac{\pi}{4}+2n\pi$ are local maximum points and $x=\frac{5\pi}{4}+2n\pi$ are local minimum points.