POWER SERIES

A power series is any series that can be written in the general form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Where *a* and c_n are numbers. The first thing to notice about a power series is that it is a function of *x*. This is different from arithmetic and geometric series where we've only dealt with numbers in the series, now there are variables in the series as well.

A power series may converge for some values of *x* and not for other values of *x*. A power series is said to converge at a point x if:

$$\lim_{p\to\infty} \sum_{n=0}^{\infty} c_n (x-a)^n$$

Exists for that specific x. A power series always converges at x=a, but convergence for the remaining values must be determined.

The most useful test for convergence of a power series is the ratio test. If $c_n \neq 0$, and if for a fixed value:

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = |x-a| L$$

Then the power series converges at that value of x if $|x-a|L<1$ and diverges if $|x-a|L>1$. If $|x-a|L=1$, then the test is inconclusive.

A power series has a unique number, R, called the radius of convergence. If:

- $|x-a| < R$ the series will converge
- $|x-a| > R$ the series will diverge

The interval of convergence is the interval of all the x's for which the power series converges.

Combing the radius of convergence and interval of convergence together, we find that:

- $a-R < x < a+R$ the power series converges
- $x < a-R \, and \, x > a+R$ the power series diverges

Concept Example:

The following problem introduces the concept reviewed within this module. Use this content as a primer for the subsequent material.

Determine the interval of convergence for the following series:

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n}$$

Solution:

Recall that the most useful test for convergence of a power series is the ratio test. If $c_n \neq 0$, and if for a fixed value:

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = |x-a| L$$

Then the power series converges at that value of x if $|x-a|L < 1$ and diverges if $|x-a|L > 1$. If $|x-a|L = 1$, then the test is inconclusive.

Analyzing the series:

$$\lim_{n\to\infty}\left|\frac{c_{n+1}\left(x-a\right)^{n+1}}{c_n\left(x-a\right)^n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{3^{n+1}}}{\frac{x^n}{3^n}}\right|=\left|x\right|\lim_{n\to\infty}\left|\frac{3^n}{3^{n+1}}\right|=\left|x\right|\frac{1}{3}$$

This tells us that the power series converges for the values of x where $\left|x\right|\frac{1}{3}<1$ and diverges for the values of x where $\left|x\right|\frac{1}{3}>1$. To determine the interval of convergence, we solve the convergence inequality and find that the power series converges on the interval:

$\left|x\right|<3$ or -3<x<3