SIMULTANEOUS LINEAR EQUATIONS

Given a number of linear equations each with a set of unknowns such as:

$$Ax_1 + Bx_2 + Cx_3 + \ldots + Zx_n = y_1$$
$$Dx_1 + Ex_2 + Fx_3 + \ldots + Yx_n = y_2$$
$$Gx_1 + Hx_2 + Ix_3 + \ldots + Xx_n = y_3$$

Where $x_1, x_2, x_3, \ldots, x_n$ are unkown values
A, B, C…Z are known coefficients and
y_1, y_2, y_3 are known quantities

Any set of equations in this format can be expressed compactly in a matrix:

$$Ax = y$$

Where x is a single column vector matrix with as many rows as there are equations.
A is a square matrix with as many columns as there are unknowns and rows as there are equations
And y is a single column matrix with as many rows as known y values

The process to solve any set of simultaneous equations is:

1. Express the set of linear equations compactly in the matrix form $Ax = y$
2. Premultiply both sides of the equation with the inverse matrix of A, A^{-1}:

   $$A^{-1}Ax = A^{-1}y$$

3. Since $A^{-1}Ax = Ix = x$, it is known then that:

   $$X = A^{-1}y$$

So as long as A^{-1} exists, all the unknown x values can be solved for. If the inverse does not exist, then the set of equations does not have a unique solution.

Concept Example:

The following problem introduces the concept reviewed within this module. Use this content as a primer for the subsequent material.

Solve the following set of linear equations:

$$-x+5y=4$$

$$2x+5y=-2$$

Solution:

The process to solve any set of simultaneous equations is:

Express the set of linear equations compactly in the matrix form $Ax=y$

$$A=\begin{bmatrix}-1 & 5\\ 2 & 5\end{bmatrix} \quad x=\begin{bmatrix}x\\ y\end{bmatrix} \qquad y=\begin{bmatrix}4\\ -2\end{bmatrix}$$

Premultiply both sides of the equation with the inverse matrix of A, A^{-1}:

$$A^{-1}Ax=A^{-1}y$$

The inverse of A is found by first swapping the leading diagonal of the 2x2 matrix and switching the signs of the other two elements such that:

$$\begin{bmatrix}5 & -5\\ -2 & -1\end{bmatrix}$$

Now find the determinate of A, which is:

$$|A| = 5(-1) - (-5)(-2) = -5 - 10 = -15$$

So the inverse is:

$$A^{-1} = -\frac{1}{15}\begin{bmatrix} 5 & -5 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{15} & \frac{1}{15} \end{bmatrix}$$

Therefore,

$$x = A^{-1}y$$

$$x = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{15} & \frac{1}{15} \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$x = \begin{bmatrix} -2 \\ \frac{2}{5} \end{bmatrix}$$

The solutions are then $x = -2$ and $y = \frac{2}{5}$

Confirm these answers by plugging them in to the original equation.

$$-x + 5y = 4$$
$$2x + 5y = -2$$

Substituting $x = 2$ and $y = \frac{2}{5}$, we get:

$$-(-2) + 5x\left(\frac{2}{5}\right) = 2 + 2 = 4$$

$$2x(-2) + 5\left(\frac{2}{5}\right) = -4 + 2 = -2$$